Int. J. Solids Structures, 1977, Vol. 13, pp. 1133–1139. Pergamon Press. Printed in Great Britain

STEADY CRACK PROPAGATION FOLLOWED BY NON-STEADY GROWTH—MODE I SOLUTION

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(Received 18 January 1977; revised 4 May 1977)

Abstract—A mode I crack moves under steady-state conditions. At some time instant the velocity of propagation changes in some arbitrary way. By use of known solutions to other elastodynamic crack problems, the stress-intensity factor for this non-steady growth is obtained. It is given in the form of convolution integrals over quantities known from the steady-state solution. For the case of a momentaneous velocity jump an explicit equation is derived. Generalization of the results to the problem of arbitrary growth after an initial self-similar propagation is outlined.

INTRODUCTION

It has become more or less generally accepted that in order to perform a correct treatment of a crack arrest problem, inertia effects must be taken into account if the velocity of the preceding crack growth has been a significant fraction of the wave speeds. In most cases such problems must be solved by aid of numerical methods such as the finite difference technique or the finite element method. However, the application of such methods to elastodynamic problems with moving cracks is by no means simple and a lot of development still remains before these methods can be used in a routine manner. In view of this, it is of interest to find analytical solutions whenever possible. Some progress has also been made for certain idealized problems.

In a recent paper[1], Nilsson treated the non-steady growth of a mode III crack, which initially propagates under steady-state conditions. Explicit formulas for the stressintensity factor could be derived in the form of convolution integrals involving quantities from the corresponding steady-state solution. The validity of these expressions is limited to a short time range after the first deviation from steady growth.

The purpose of the present paper is to derive corresponding results for the mode I problem. The method of solution will be very similar to the one used in [1]. To this end we will employ previously presented results for the case when the crack tip stops momentaneously after a steady growth (Nilsson[2]). Almost simultaneously with [2], Achenbach and Tolikas[3] presented solutions to a similar problem. They considered instantaneous velocity changes to a non-zero constant speed and the behaviour immediately after the jump was examined. Furthermore we will need the generalizations of Freunds's solutions[4–7] that have recently been given by Kostrov[8] and Burridge[9]. In these papers the non-steady growth of an initially stationary crack is considered for general time dependent loading.

STATEMENT OF THE PROBLEM AND SOME BASIC RESULTS

Consider (Fig. 1) a mode I crack with traction free crack surfaces propagating under steady-state conditions, i.e. the state with respect to a coordinate-system (η, ξ) attached to the tip is independent of time. Let the crack propagation velocity be V_1. We denote this problem as (A) and assume that its solution is known, in particular the normal stress in front of the crack tip $p^s(\eta)$ and the displacement of the crack surface $w^s(\eta)$.

At $t = 0$, corresponding to $x = 0$ with respect to a fixed (x, y)-system, the crack tip velocity now changes in some arbitrary way. We denote the position of the tip by $a(t)$, which is the tip's coordinate with respect to (x, y). The purpose is now to determine the stressintensity factor $K_I(t)$ defined by

$$K_I = \lim_{x \to +a(t)} (2\pi(x - a(t)))^{1/2} \sigma_y(x, t), \quad y = 0 \tag{1}$$

for this problem during the time interval $0 < t < t_d$. t_d is the time instant at which a disturbance

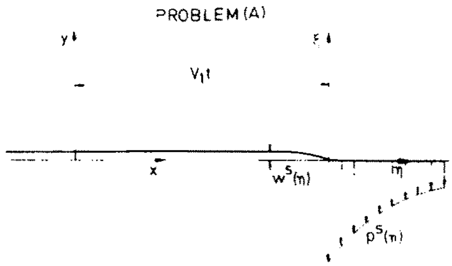

Fig. 1. The steady-state problem.

emitted from the tip at $t = 0$, hits the tip again after having been reflected at the boundary or at the other crack tip.

Suppose now that the crack tip stops instantaneously at $t = 0$ (problem B). We will then obtain some time dependent stress-distribution ahead of the tip $\sigma_y = p^{st}(x, t)$ (Fig. 2). The stress-intensity factor for this case was given in [2]. After some changes of notation, guided by the results of [3], we have

$$K_I^{st}(t) = \frac{(c + d_1)S_+(d_1, \infty)}{(a + d_1)^{1/2}d_1^{1/2}} 2\mu(1 - a^2/b^2)\left(\frac{2}{\pi}\right)^{1/2} z(t) \tag{2}$$

$$z(t) = \int_{\eta=0}^{-t/d_1} (t/d_1 + \eta)^{-1/2}\, \mathrm{d}w^s(\eta) \tag{3}$$

where

$$S_\pm(\lambda, d) = \exp\left[-\frac{1}{\pi}\int_{a_+}^{b_\pm} \tan^{-1}(q(\kappa, d))\frac{d\kappa}{\kappa \pm \lambda}\right] \tag{4}$$

$$q(\kappa, d) = \frac{4\kappa^2|\alpha||\beta|}{(2\kappa^2 - b^2 - b^2\kappa^2/d^2 \mp 2b^2\kappa/d)^2} \tag{5}$$

$$a_\pm = \frac{a}{1 \mp a/d} \tag{6}$$

$$b_\pm = \frac{b}{1 \mp b/d} \tag{7}$$

$$\alpha^2 = a^2(1 - \lambda/d)^2 - \lambda^2 \tag{8}$$

$$\beta^2 = b^2(1 - \lambda/d)^2 - \lambda^2 \tag{9}$$

$$a = 1/C_1 \tag{10}$$

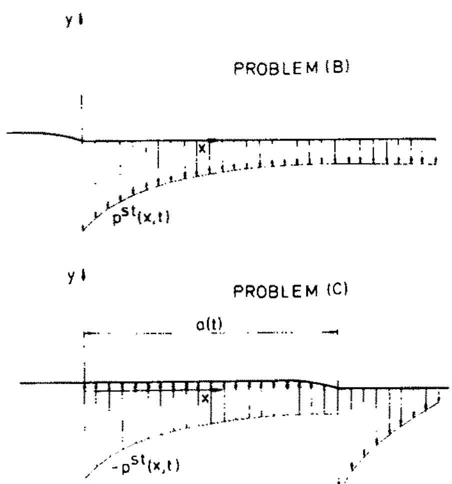

Fig. 2. Illustration of the stopping crack (B) and the arbitrary growth (C).

$$b = 1/C_2 \tag{11}$$

$$c = 1/C_R \tag{12}$$

$$d_1 = 1/V_1. \tag{13}$$

C_1 is the velocity of irrotational waves, C_2 the velocity of the equivoluminal waves, C_R the Rayleigh wave velocity and μ the shear-modulus. These results (eqns (2–3)) can also be derived from the equation derived in the Appendix of [3]. However, as previously mentioned these authors limit their discussion to the K_I-value immediately after the velocity jump.

We now consider the following boundary value problem (C) for a half-space. The medium is at rest for $t \leq 0$.

$$\sigma_y = -H(x)H(t)p^{st}(x, t) \quad x < a(t) \tag{14}$$

$$w = 0 \qquad x > a(t). \tag{15}$$

H denotes the unit step function. Superposing (C) onto problem (B), we obviously obtain the boundary conditions for an arbitrary crack growth $a(t)$ for $t > 0$. Since the only contribution to the stress-intensity factor comes from problem (C), we need only evaluate $K_I(t)$ for the loading given by eqns (14)–(15). This is in fact the problem considered in Refs. [8–9]. From [9] we have

$$K_I(t) = -f_F(\dot{a}) \iint_D H(x)H(\tau)p^{st}(x, \tau)g(a(t) - x, t - \tau)\,\mathrm{d}x\,\mathrm{d}\tau \tag{16}$$

where

$$f_F(\dot{a}) = \frac{1 - c/d_2}{(1 - a/d_2)^{1/2}S_+(d_2, d_2)} \tag{17}$$

$$d_2 = 1/\dot{a}(t) \tag{18}$$

$$D:\ 0 < \tau < t \tag{19}$$

$$a(t) - (t - \tau)/a < x < a(t) - (t - \tau)/c \tag{20}$$

$g(x, t)$ is a complicated function given by Burridge[9]. Its particular form is not needed here. One important feature of eqn (16) is that the motion of the crack tip enters the integral only through $a(t)$. This means that the integral will have the same value for any growth history with the property that the position of the crack tip is $a(t)$ at the considered time instant. Since it is difficult, although in principle possible, to obtain the stress distribution p^{st} and performing the integration in (14), we will instead consider two particular crack growth histories with this property. For the so chosen motions it is possible to obtain $K_I(t)$ by simpler methods and thereby evaluate the integral in eqn (14) indirectly.

We shall firstly, however, rewrite eqn (2) in a somewhat more convenient form. To this end we use the following identities for $S_+(\lambda, d)$ (Freund[4, 6]).

$$S_+(\lambda, d)S_-(\lambda, d)(\lambda - c_+)(\lambda - d)^2R(d, \infty) = R(\lambda, d)d^4 \tag{21}$$

$$S_+(\lambda, d) = S_-(1/(1/d - 1/\lambda), \infty)/S_-(d, \infty) \tag{22}$$

where

$$R(\lambda, d) = 4\lambda^2\alpha\beta + (2\lambda^2 - b^2 - b^2\lambda^2/d^2 + 2b^2\lambda/d)^2 \tag{23}$$

$$c_\pm = \frac{c}{1 \mp c/d}. \tag{24}$$

Using (2), (17) and (21)–(24) we obtain

$$K_I^{st}(t) = \frac{1}{f_F(V_1)} \int_{\eta=0}^{-t/d_1} \frac{4}{\pi L(V_1)(\eta + t/d_1)^{1/2}}\,\mathrm{d}w^s(\eta) \tag{25}$$

where

$$L(V_1) = (2/\pi)^{1/2} \frac{b^2 d_1^2 (1 - a^2/d_1^2)^{1/2}}{\mu R(d, \infty)}. \tag{26}$$

In the limit when t goes to zero, only the singular part of $w^s(\eta)$ contributes to $K_I(t)$. If K_I^s is the stress-intensity factor for the steady-state problem, we have

$$w^s(\eta) \to (-\eta)^{1/2} L(V_1) K_I^s \quad \text{as } \eta \to -0. \tag{27}$$

Insertion of (27) into (25) then yields the result

$$\underset{\lim t \to +0}{K_I^{st}(t)} = (1/f_F(V_1)) K_I^s \tag{28}$$

which is in agreement with what was obtained in Ref. [3].

SOLUTION FOR THE CASE $a(t) < V_1 t$

Following the method used in [1], we consider the crack growth history defined by

$$\dot{a}_e(\tau) = V_1 \quad \text{for } 0 < \tau < a(t)/V_1 < t \tag{29}$$

$$\dot{a}_e(\tau) = 0 \quad \text{for } a(t)/V_1 < \tau \le t. \tag{30}$$

This motion obviously satisfies the necessary requirements. It is the case when the crack tip continues to move steadily and instead stops at the point $x = a(t)$. $K_I(t)$ is then given by eqn (25) with t replaced by $t - d_1 a(t)$. $K_I(t)$ for a general crack growth history is obtained by multiplying with the velocity dependent factor of eqn (16), i.e.

$$K_I(t) = \frac{f_F(\dot{a})}{f_F(V_1)} \int_{\eta=0}^{\Delta(t)} \frac{4}{\pi L(V_1)(\eta - \Delta(t))^{1/2}} \mathrm{d}w^s(\eta) \tag{31}$$

$$\Delta(t) = a(t) - V_1 t. \tag{32}$$

This equation is completely analogous to eqn (12) of Ref. [1]. It is also easily verified that in the case $\dot{a}(t)$ is constant, eqn (31) is equivalent to result derived in [3].

SOLUTION FOR THE CASE $a(t) = V_2 T$

Before considering the general case $a(t) > V_1 t$, we shall give attention to a particular motion of this kind. Suppose that for t larger than zero the position of the crack tip is given by $a(t) = V_2 t$, where V_2 is a constant larger than V_1. In analogy with Ref. [4] we consider the following elementary problem. Introduce a second moving coordinate system (x', y') attached to the crack tip moving with velocity V_2 (Fig. 3). A concentrated force of unit magnitude appears at $x = \eta = 0$ at zero time t and then remains fixed with respect to the (η, ξ)-system. The boundary conditions take the form

$$y = 0: \quad \sigma_{xy} = 0 \tag{33}$$

$$\sigma_y = -\delta(x' - (V_2 - V_1)t) H(t) \quad x' > 0 \tag{34}$$

$$w = 0 \quad x' < 0 \tag{35}$$

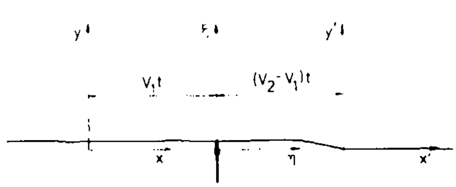

Fig. 3. The elementary problem.

where δ denotes Dirac's unit impulse function. This problem is a special case of the fundamental problem solved by Freund in Ref. [6], where a point force with a linear time-dependence was considered. Setting $m = 0$ in eqn (3.6) of his paper we obtain for the stress-intensity factor of the present elementary problem

$$K_I^e(t) = (2/\pi)^{1/2} t^{-1/2} \frac{(a + d_{12}(1 - a/d_2))^{1/2} d_{12}}{(1 - a/d_2)^{1/2} S_+(d_{12}, d_2)(d_{12} + c_+)} \tag{36}$$

$$\begin{aligned} d_{12} &= 1/(V_2 - V_1) \\ d_2 &= 1/V_2. \end{aligned} \tag{37}$$

The problem of the constant velocity crack which suddenly increases its velocity can now be represented in the following way. As the crack tip moves from η to $\eta + \mathrm{d}\eta$ with respect to the (η, ξ)-system, this is equivalent to the application of a concentrated force of magnitude $p^s(\eta)\,\mathrm{d}\eta$ to the crack surface. If we then let this force move with velocity V_1 the steady-state stress distribution is negated over the length increment $\mathrm{d}\eta$. Considering an ensemble of such elementary problems with η ranging from zero to $t(V_2 - V_1)$, we find after an integration that the steady-state stress distribution is exactly negated over the appropriate segment. The contribution of a particular one of these forces to the total stress-intensity factor is $K_I^e(t - d_{12}\eta)p^s(\eta)\,\mathrm{d}\eta$. Integrating over all such contributions we have

$$K_I(t) = d_{12}^{-1/2} \int_{\eta=0}^{t/d_{12}} p^s(\eta) K^e(t/d_{12} - \eta)\,\mathrm{d}\eta. \tag{39}$$

SOLUTION FOR THE CASE $a(t) > V_1 t$

Using the same strategy as earlier, we construct a particular crack motion as

$$\dot{a}_e(t) = V_2 \qquad V_2 = a(t)/t > V_1. \tag{40}$$

This motion has the property that the tip's position is $a(t)$ at the time instant t. In order to take the actual velocity into account, we multiply eqn (39) by $f_F(\dot{a}(t))/f_F(V_2)$. Insertion of (40) into (39) and using eqns (21) and (36) yield the following result for a general crack tip motion

$$K_I(t) = \frac{f_F(\dot{a})}{f_F(V_1)} \int_{\eta=0}^{\Delta(t)} \left(\frac{2}{\pi}\right)^{1/2} (\Delta(t) - \eta)^{-1/2} p^{st}(\eta)\,\mathrm{d}\eta \tag{41}$$

with $\Delta(t)$ defined by eqn (32).

This is also in analogy to what was found for the mode III problem. Equations (31) and (41) now constitute the complete solution to the title problem. The results can be summarized as

$$K_I(t) = (f_F(\dot{a})/f_F(V_1))h(\Delta(t); V_1). \tag{42}$$

The motion of the crack tip enters only through $\dot{a}$ in the velocity dependent factor and through $\Delta(t)$ into the function h, which can be calculated once and for all for a specific problem. This can be interpreted in the following way. K_I depends only on the position of the tip with respect to the moving (η, ξ)-system and on tip velocity relative to the same system. The preceding motion enters in form of the parameter V_1. We note the analogy with the case when $V_1 = 0$.

In the limiting case as $\Delta(t)$ tends to zero, only the singular part of $p^s(\eta)$ gives contributions to the integral. Insertion of eqn (1) into (41) yields a result which is also obtained from (31) as $\Delta(t)$ goes to zero. Thus at a discontinuous velocity change we have

$$\frac{K_I^+}{K_I^-} = \frac{f_F(V^+)}{f_F(V^-)}. \tag{43}$$

The plus-sign here indicates quantities immediately after the velocity jump and the minus-sign immediately before the jump. It can be shown that this particular result is not limited by the previous restriction that the crack propagation preceding the jump should be steady-state, since at very small times after the jump only the singular parts of the field quantities contribute and these have the same structure whether the crack is moving steadily or not. This fact was utilized in Ref. [3].

GENERALIZATION TO SELF-SIMILAR PROBLEMS

One may ask what can be done if the assumption of steady-state growth is abandoned. Suppose that we have a general problem. If the crack tip is stopped at a certain time instant, some time dependent stress-distribution will result in front of the crack. If this stress-distribution can be calculated, we can by insertion into eqn (16) obtain $K_I(t)$ during the time range before secondary diffraction effects set in For the problem considered above we were successful with this scheme, since sufficiently simple relations such as eqns (25) and (39) could be derived.

There is at least one other class of dynamic crack propagation problems for which a similar treatment is possible. This is the self-similar problems of which the uniformly expanding crack (Fig. 4) firstly considered by Broberg[10] may serve as a typical example.

We focus the attention to right crack tip. The position is $x = V_1 t$. The important feature of this problem is that the particle velocities and the stresses are homogeneous functions of degree zero in the spatial coordinates and time. In particular, denote the displacement rate of the upper crack surface by $\dot{w}^{ss}(\omega = x/t)$. Suppose now that when the tip reaches $x = x_0$, the velocity changes in some arbitrary way so that the position of the is $x = a(t)$.

If we consider the case when $a(t) < V_1 t$, it is obvious that precisely the same method as for the steady-state problem can be used, provided that a relation for a stopping crack analogous to eqn (25) can be found. This problem has indeed been treated by Freund[11] using a scheme of superposition of solutions to an elementary problem involving motion of a velocity dislocation. For a crack which stops instantaneously at x_0 the following relation was derived

$$K_I^{st}(t) = \int_{\omega = x_0/t}^{V_1} f_D(\omega)(t - x_0/\omega)^{1/2}\,\mathrm{d}\dot{w}^{ss}(\omega) \tag{44}$$

where

$$f_D(\omega) = \frac{(c+\omega)S_+(\omega,\infty)}{(a+\omega)^{1/2}}4\mu(1 - a^2/b^2)(2/\pi)^{1/2}. \tag{45}$$

It is now a simple matter to obtain $K_I(t)$ for a general crack growth history in analogy with the previous analysis. Using the same arguments as for the steady-state problem we get

$$K_I(t) = f_F(\dot{a})\int_{\omega = a(t)/t}^{V_1} f_D(\omega)(t - a(t)/\omega)^{1/2}\,\mathrm{d}\dot{w}^{ss}(\omega). \tag{46}$$

This is the analogy to eqn (31) for the present case. Again it is only valid for times smaller than t_d.

In order to solve the case when $a(t) > V_1 t$ a relation corresponding to eqn (37) must be derived. This can presumably be done in a fairly straight forward manner by aid of the elementary problem discussed previously. We will not pursue the analysis further since this problem is of limited practical interest.

The relation (46) is clearly valid for any problem where $\dot{w}^{ss}$ is a homogenous function of

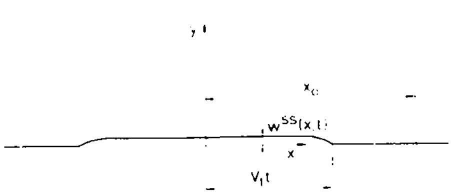

Fig. 4. The uniformly expanding crack.

degree zero in x and t. If some other time derivative of the displacement is a homogeneous function of degree zero, corresponding relations can be derived which will contain the appropriate time derivate instead of the particle velocity.

CONCLUDING REMARKS

Most of the until today solved dynamic crack propagation problems fall into one of the categories steady-state or self-similar crack growth. We can therefore expect that derived results will have a rather wide range of applications. The main limitation is that the results are only valid during a fairly short time range after the deviation from constant velocity growth has taken place. In many cases, however, the retardation time before arrest is quite short and may well be contained within the time interval $t < t_d$.

The application of analogous mode III results to crack arrest problems was discussed in Ref. [1]. The same approach is evidently valid also in the mode I case and the reader is referred to [1] for a discussion of these aspects.

As can be seen the function $f_F(V)$ is quite complicated. To this end we cite a useful approximation derived by Rose[12] of the form

$$f_F(V) \approx (1 - cV)(1 - \gamma V)^{-1/2} \tag{47}$$

where γ is a constant, slightly dependent on Poisson's ratio. With $\gamma = 0.948b$, $f_F(V)$ is according to Rose approximated to within two percent over the entire velocity range and all values of Poisson's ratio. This is sufficiently accurate for all practical purposes.

Acknowledgement—This work was performed during the tenure of a grant from the Swedish Board of Technical Development (STU). The author wants to express his gratitude for this support.

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